CHAPTER 38
VIBRATION AND CONTROL OF VIBRATION

T. S. Sankar, Ph.D., Eng.
Professor and Chairman
Department of Mechanical Engineering
Concordia University
Montreal, Quebec, Canada

R. B. Bhat, Ph.D.
Associate Professor
Department of Mechanical Engineering
Concordia University
Montreal, Quebec, Canada

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38.1 INTRODUCTION

Vibration analysis and control of vibrations are important and integral aspects of every machine design procedure. Establishing an appropriate mathematical model, its analysis, interpretation of the solutions, and incorporation of these results in the design, testing, evaluation, maintenance, and troubleshooting require a sound understanding of the principles of vibration. All the essential materials dealing with various aspects of machine vibrations are presented here in a form suitable for most design applications. Readers are encouraged to consult the references for more details.

38.2 SINGLE-DEGREE-OF-FREEDOM SYSTEMS

38.2.1 Free Vibration

A single-degree-of-freedom system is shown in Fig. 38.1. It consists of a mass m constrained by a spring of *stiffness* k, and a damper with *viscous damping coefficient* c. The stiffness coefficient k is defined as the spring force per unit deflection. The coef-

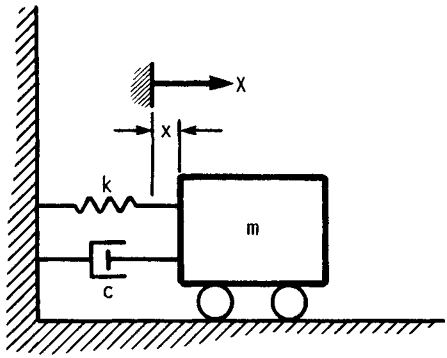

FIGURE 38.1 Representation of a single-degree-of-freedom system.

ficient of viscous damping c is the force provided by the damper opposing the motion per unit velocity.

If the mass is given an initial displacement, it will start vibrating about its equilibrium position. The equation of motion is given by

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{38.1}$$

where x is measured from the equilibrium position and dots above variables represent differentiation with respect to time. By substituting a solution of the form $x = e^{st}$ into Eq. (38.1), the characteristic equation is obtained:

$$ms^2 + cs + k = 0 \tag{38.2}$$

The two roots of the characteristic equation are

$$s = \zeta\omega_n \pm i\omega_n(1 - \zeta^2)^{1/2} \tag{38.3}$$

where $\omega_n = (k/m)^{1/2}$ is undamped *natural frequency*
$\zeta = c/c_c$ is *damping ratio*
$c_c = 2m\omega_n$ is *critical damping coefficient*
$i = \sqrt{-1}$

Depending on the value of ζ, four cases arise.

Undamped System ($\zeta = 0$). In this case, the two roots of the characteristic equation are

$$s = \pm i\omega_n = \pm i(k/m)^{1/2} \tag{38.4}$$

and the corresponding solution is

$$x = A \cos \omega_n t + B \sin \omega_n t \tag{38.5}$$

where A and B are arbitrary constants depending on the initial conditions of the motion. If the initial displacement is x_0 and the initial velocity is v_0, by substituting these values in Eq. (38.5) it is possible to solve for constants A and B. Accordingly, the solution is

$$x = x_0 \cos \omega_n t + \frac{v_0}{\omega_n} \sin \omega_n t \tag{38.6}$$

Here, ω_n is the natural frequency of the system in radians per second (rad/s), which is the frequency at which the system executes free vibrations. The *natural frequency* is

$$f_n = \frac{\omega_n}{2\pi} \tag{38.7}$$

where f_n is in cycles per second, or hertz (Hz). The *period* for one oscillation is

$$\tau = \frac{1}{f_n} = \frac{2\pi}{\omega_n} \tag{38.8}$$

The solution given in Eq. (38.6) can also be expressed in the form

$$x = X \cos (\omega_n - \theta) \tag{38.9}$$

where

$$X = \left[x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2\right]^{1/2} \qquad \theta = \tan^{-1} \frac{v_0}{\omega_n x_0} \tag{38.10}$$

The motion is harmonic with a *phase angle* θ as given in Eq. (38.9) and is shown graphically in Fig. 38.4.

Underdamped System (0 < ζ < 1). When the system damping is less than the critical damping, the solution is

$$x = [\exp(-\zeta\omega_n t)]\,(A \cos \omega_d t + B \sin \omega_d t) \tag{38.11}$$

where

$$\omega_d = \omega_n(1 - \zeta^2)^{1/2} \tag{38.12}$$

is the *damped natural frequency* and A and B are arbitrary constants to be determined from the initial conditions. For an initial amplitude of x_0 and initial velocity v_0,

$$x = [\exp(-\zeta\omega_n t)]\left(x_0 \cos \omega_d t + \frac{\zeta\omega_n x_0 + v_0}{\omega_d} \sin \omega_d t\right) \tag{38.13}$$

which can be written in the form

$$x = [\exp(-\zeta\omega_n t)]\, X \cos(\omega_d t - \theta)$$

$$X = \left[x_0^2 + \left(\frac{\zeta\omega_n x_0 + v_0}{\omega_d}\right)^2\right]^{1/2} \tag{38.14}$$

and

$$\theta = \tan^{-1} \frac{\zeta\omega_n x_0 + v_0}{\omega_d}$$

An underdamped system will execute exponentially decaying oscillations, as shown graphically in Fig. 38.2.

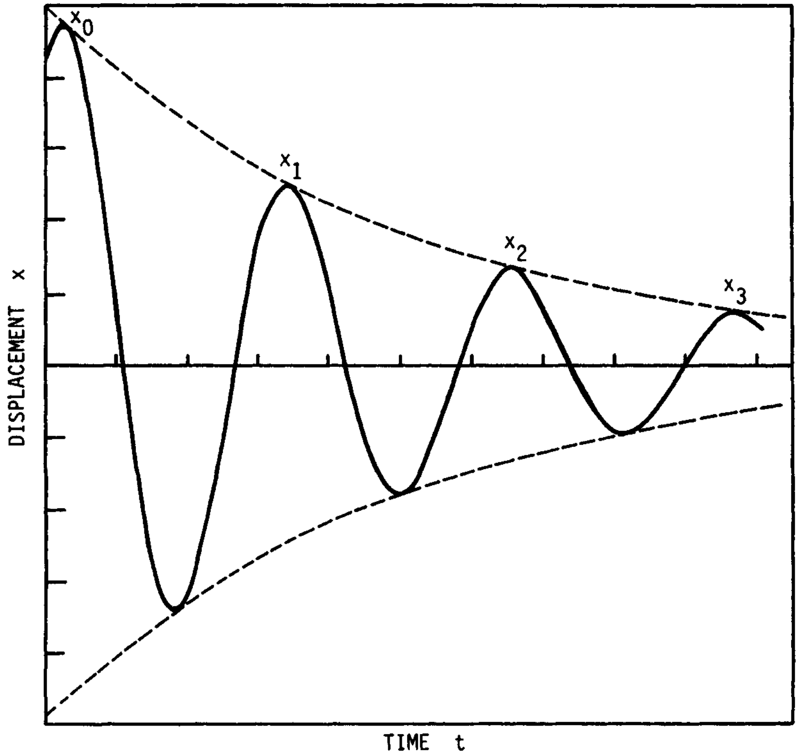

FIGURE 38.2 Free vibration of an underdamped single-degree-of-freedom system.

The successive maxima in Fig. 38.2 occur in a periodic fashion and are marked $X_0, X_1, X_2, \ldots$ The ratio of the maxima separated by n cycles of oscillation may be obtained from Eq. (38.13) as

$$\frac{X_n}{X_0} = \exp(-n\delta) \tag{38.15}$$

where

$$\delta = \frac{2\pi\zeta}{(1-\zeta^2)^{1/2}}$$

is called the *logarithmic decrement* and corresponds to the ratio of two successive maxima in Fig. 38.2. For small values of damping, that is, $\zeta << 1$, the logarithmic decrement can be approximated by

$$\delta = 2\pi\zeta \tag{38.16}$$

Using this in Eq. (38.14), we find

$$\frac{X_n}{X_0} = \exp(-2\pi n\zeta) \simeq 1 - 2\pi n\zeta \tag{38.17}$$

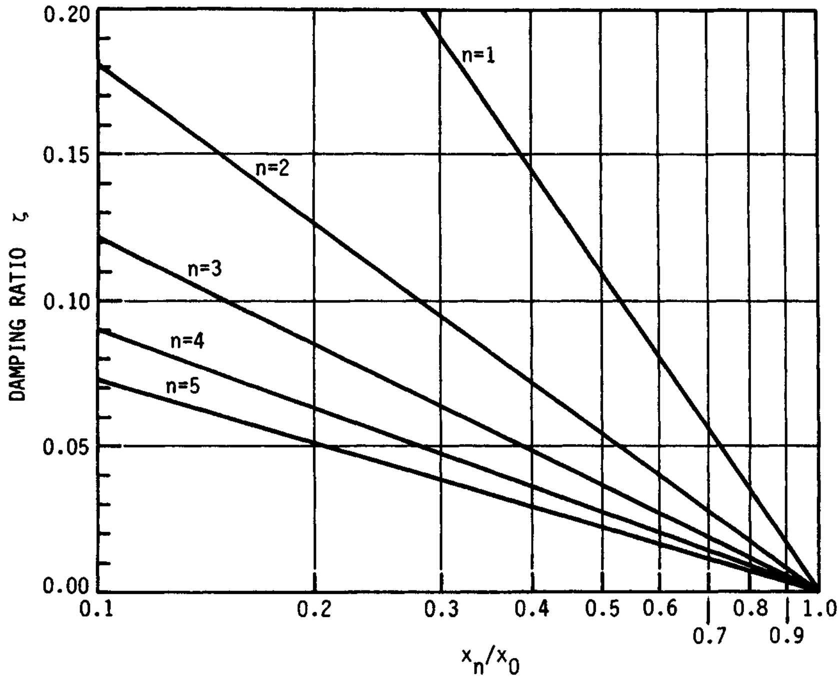

FIGURE 38.3 Variation of the ratio of displacement maxima with damping.

The equivalent viscous damping in a system is measured experimentally by using this principle. The system at rest is given an impact which provides initial velocity to the system and sets it into free vibration. The successive maxima of the ensuing vibration are measured, and by using Eq. (38.17) the damping ratio can be evaluated. The variation of the decaying amplitudes of free vibration with the damping ratio is plotted in Fig. 38.3 for different values of n.

Critically Damped System ($\zeta = 1$). When the system is critically damped, the roots of the characteristic equation given by Eq. (38.3) are equal and negative real quantities. Hence, the system does not execute oscillatory motion. The solution is of the form

$$x = (A + Bt) \exp(-\omega_n t) \tag{38.18}$$

and after substitution of initial conditions,

$$x = [x_0 + (v_0 + x_0\omega_n)t] \exp(-\omega_n t) \tag{38.19}$$

This motion is shown graphically in Fig. 38.4, which gives the shortest time to rest.

Overdamped System ($\zeta > 1$). When the damping ratio ζ is greater than unity, there are two distinct negative real roots for the characteristic equation given by Eq. (38.3). The motion in this case is described by

$$x = \exp(-\zeta\omega_n t)\,[A \exp \omega_n t\sqrt{\zeta^2 - 1} + B \exp(-\omega_n t\sqrt{\zeta^2 - 1})] \tag{38.20}$$

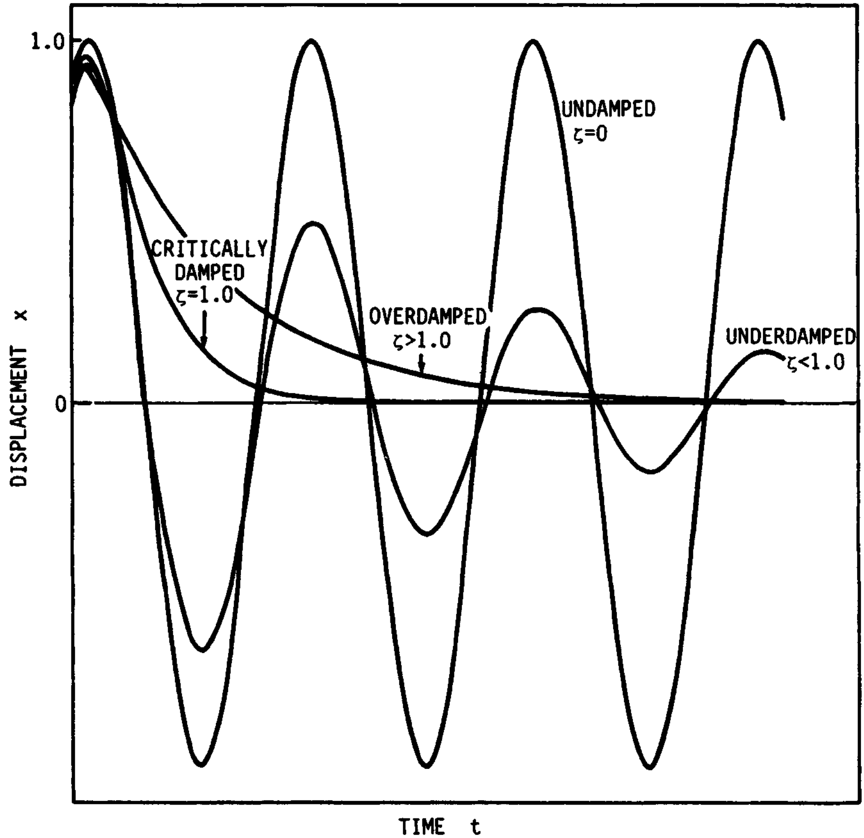

FIGURE 38.4 Free vibration of a single-degree-of-freedom system under different values of damping.

where

$$A = \frac{1}{2}\left(x_0 + \frac{v_0 + \zeta\omega_n x_0}{\omega_n}\right) \qquad B = \frac{1}{2}\left(\frac{x_0 + \zeta\omega_n x_0}{\omega_0}\right)$$

and

$$\omega_0 = \omega_n\sqrt{\zeta^2 - 1}$$

All four types of motion are shown in Fig. 38.4.

If the mass is suspended by a spring and damper as shown in Fig. 38.5, the spring will be stretched by an amount δ_{st}, the static deflection in the equilibrium position. In such a case, the equation of motion is

$$m\ddot{x} + c\dot{x} + k(x + \delta_{st}) = mg \tag{38.21}$$

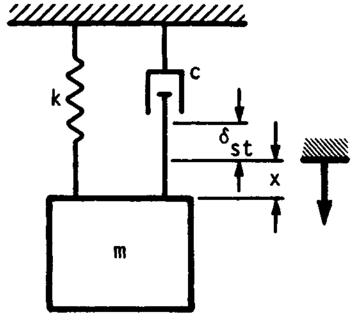

FIGURE 38.5 Model of a single-degree-of-freedom system showing the static deflection due to weight.

Since the force in the spring due to the static equilibrium is equal to the weight, or $k\delta_{st} = mg = W$, the equation of motion reduces to

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{38.22}$$

which is identical to Eq. (38.1). Hence the solution is also similar to that of Eq. (38.1). In view of Eq. (38.21) and since $\omega_n = (k/m)^{1/2}$, the natural frequency can also be obtained by

$$\omega_n = \left(\frac{g}{\delta_{st}}\right)^{1/2} \tag{38.23}$$

An approximate value of the fundamental natural frequency of any complex mechanical system can be obtained by reducing it to a single-degree-of-freedom system. For example, a shaft supporting several disks (wheels) can be reduced to a single-degree-of-freedom system by lumping the masses of all the disks at the center and obtaining the equivalent stiffness of the shaft by using simple flexure theory.

38.2.2 Torsional Systems

Rotating shafts transmitting torque will experience torsional vibrations if the torque is nonuniform, as in the case of an automobile crankshaft.

In rotating shafts involving gears, the transmitted torque will fluctuate because of gear-mounting errors or tooth profile errors, which will result in torsional vibration of the geared shafts.

A single-degree-of-freedom torsional system is shown in Fig. 38.6. It has a massless shaft of torsional stiffness k, a damper with damping coefficient c, and a disk with polar mass moment of inertia J. The torsional stiffness is defined as the resisting torque of the shaft per unit of angular twist, and the damping coefficient is the resisting torque of the damper per unit of angular velocity. Either the damping can be externally applied, or it can be inherent structural damping. The equation of motion of the system in torsion is given

$$J\ddot{\theta} + c\dot{\theta} + k\theta = 0 \tag{38.24}$$

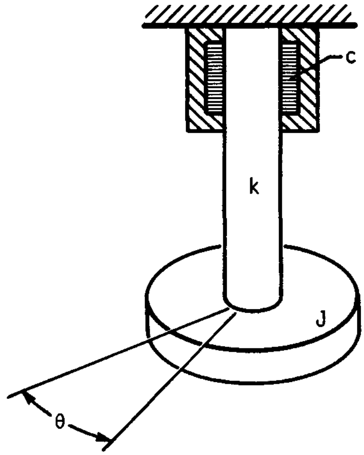

FIGURE 38.6 A representation of a one-freedom torsional system.

Equation (38.24) is in the same form as Eq. (38.1), except that the former deals with moments whereas the latter deals with forces. The solution of Eq. (38.24) will be of the same form as that of Eq. (38.1), except that J replaces m and k and c refer to torsional stiffness and torsional damping coefficient.

38.2.3 Forced Vibration

System Excited at the Mass. A vibrating system with a sinusoidal force acting on the mass is shown in Fig. 38.7. The equation of motion is

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t \tag{38.25}$$

Assuming that the steady-state response lags behind the force by an angle θ, we see that the solution can be written in the form

$$x_s = X \sin (\omega t - \theta) \tag{38.26}$$

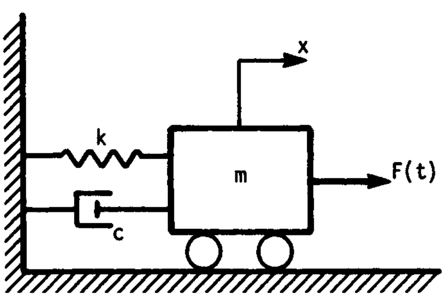

FIGURE 38.7 Oscillating force $F(t)$ applied to the mass.

Substituting in Eq. (38.26), we find that the steady-state solution can be obtained:

$$x_s = \frac{(F_0/k)\sin(\omega t - \theta)}{[(1 - \omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2]^{1/2}} \tag{38.27}$$

Using the complementary part of the solution from Eq. (38.19), we see that the complete solution is

$$x = x_s + \exp(-\zeta\omega_n t)\,[A\exp(\omega_n t\sqrt{\zeta^2 - 1}) + B\exp(-\omega_n t\sqrt{\zeta^2 - 1})] \tag{38.28}$$

If the system is undamped, the response is obtained by substituting $c = 0$ in Eq. (38.25) or $\zeta = 0$ in Eq. (38.28). When the system is undamped, if the exciting frequency coincides with the system natural frequency, say $\omega/\omega_n = 1.0$, the system response will be infinite. If the system is damped, the complementary part of the solution decays exponentially and will be nonexistent after a few cycles of oscillation; subsequently the system response is the steady-state response. At steady state, the nondimensional response amplitude is obtained from Eq. (38.27) as

$$\frac{X}{F_0/k} = \left[\left(\frac{1 - \omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2\right]^{-1/2} \tag{38.29}$$

and the phase between the response and the force is

$$\theta = \tan^{-1}\frac{2\zeta\omega/\omega_n}{1 - \omega^2/\omega_n^2} \tag{38.30}$$

When the forcing frequency ω coincides with the damped natural frequency ω_d, the response amplitude is given by

$$\frac{X_{\max}}{F_0/k} = \frac{1}{\zeta(4 - 3\zeta^2)^{1/2}} \tag{38.31}$$

The maximum response or resonance occurs when $\omega = \omega_n(1 - 2\zeta^2)^{1/2}$ and is

$$\frac{X_{\max}}{F_0/k} = \frac{1}{2\zeta(1 - \zeta^2)^{1/2}} \tag{38.32}$$

For structures with low damping, ω_d approximately equals ω_n, and the maximum response is

$$\frac{X_{\max}}{F_0/k} = \frac{1}{2\zeta} \tag{38.33}$$

The response amplitude in Eq. (38.29) is plotted against the forcing frequency in Fig. 38.8. The curves start at unity, reach a maximum in the neighborhood of the system natural frequency, and decay to zero at large values of the forcing frequency. The response is larger for a system with low damping, and vice versa, at any given frequency. The phase difference between the response and the excitation as given in Eq. (38.30) is plotted in Fig. 38.9. For smaller forcing frequencies, the response is nearly in phase with the force; and in the neighborhood of the system natural frequency, the response lags behind the force by approximately 90°. At large values of forcing frequencies, the phase is around 180°.

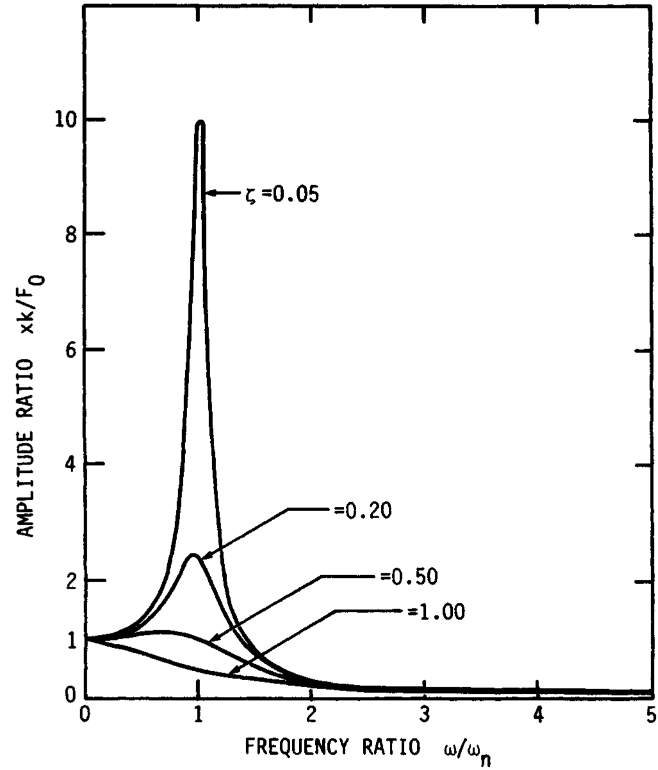

FIGURE 38.8 Displacement-amplitude frequency response due to oscillating force.

Steady-State Velocity and Acceleration Response. The steady-state velocity response is obtained by differentiating the displacement response, given by Eq. (38.27), with respect to time:

$$\frac{\dot{x}_s}{F_0\omega_n/k} = \frac{\omega/\omega_n}{[(1-\omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2]^{1/2}} \tag{38.34}$$

And the steady-state acceleration response is obtained by further differentiation and is

$$\frac{\ddot{x}_s}{F_0\omega_n^2/k} = \frac{(\omega/\omega_n)^2}{[(1-\omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2]^{1/2}} \tag{38.35}$$

These are shown in Figs. 38.10 and 38.11 and also can be obtained directly from Fig. 38.8 by multiplying the amplitude by ω/ω_n and $(\omega/\omega_n)^2$, respectively.

Force Transmissibility. The force F_T transmitted to the foundation by a system subjected to an external harmonic excitation is

$$F_T = c\dot{x} + kx \tag{38.36}$$

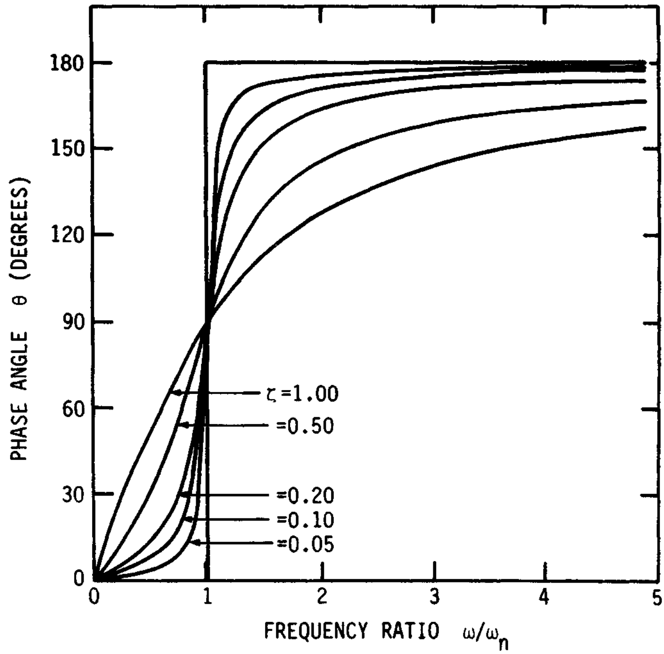

FIGURE 38.9 Phase-angle frequency response for forced motion.

Substituting the system response from Eq. (38.27) into Eq. (38.36) gives

$$\frac{F_T}{F_0} = T \sin(\omega t - \theta) \tag{38.37}$$

where the nondimensional magnitude of the transmitted force T is given by

$$T = \left[\frac{1 + (2\zeta\omega/\omega_n)^2}{(1 - \omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2}\right]^{1/2} \tag{38.38}$$

and the phase between F_T and F_0 is given by

$$\theta = \tan^{-1} \frac{2\zeta(\omega/\omega_n)^3}{1 - \omega^2/\omega_n^2 + 4\zeta^2\omega^2/\omega_n^2} \tag{38.39}$$

The *transmissibility* T is shown in Fig. 38.12 versus forcing frequency. At very low forcing frequencies, the transmissibility is close to unity, showing that the applied force is directly transmitted to the foundation. The transmissibility is very large in the vicinity of the system natural frequency, and for high forcing frequencies the transmitted force decreases considerably. The phase variation between the transmitted force and the applied force is shown in Fig. 38.13.

Rotating Imbalance. When machines with rotating imbalances are mounted on elastic supports, they constitute a vibrating system subjected to excitation from the

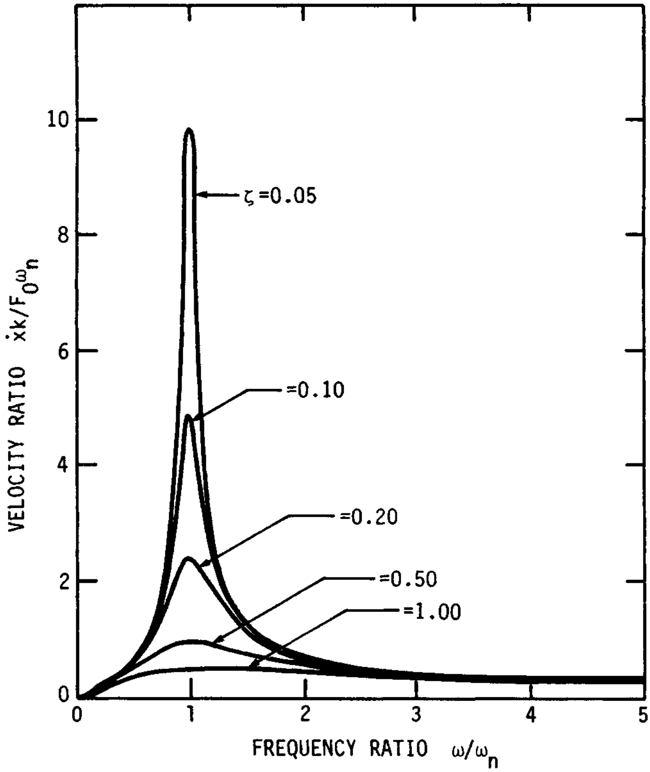

FIGURE 38.10 Velocity frequency response.

rotating imbalance. If the natural frequency of the system coincides with the frequency of rotation of the machine imbalance, it will result in severe vibrations of the machine and the support structure.

Consider a machine of mass M supported as shown in Fig. 38.14. Let the imbalance be a mass m with an eccentricity e and rotating with a frequency ω. Consider the motion x of the mass $M - m$, with x_m as the motion of the unbalanced mass m relative to the machine mass M. The equation of motion is

$$(M - m)\ddot{x} + m(\ddot{x} + \ddot{x}_m) + c\dot{x} + kx = 0 \tag{38.40}$$

The motion of the unbalanced mass relative to the machine is

$$x_m = e \sin \omega t \tag{38.41}$$

Substitution in Eq. (38.40) leads to

$$M\ddot{x} + c\dot{x} + kx = me\omega^2 \sin \omega t \tag{38.42}$$

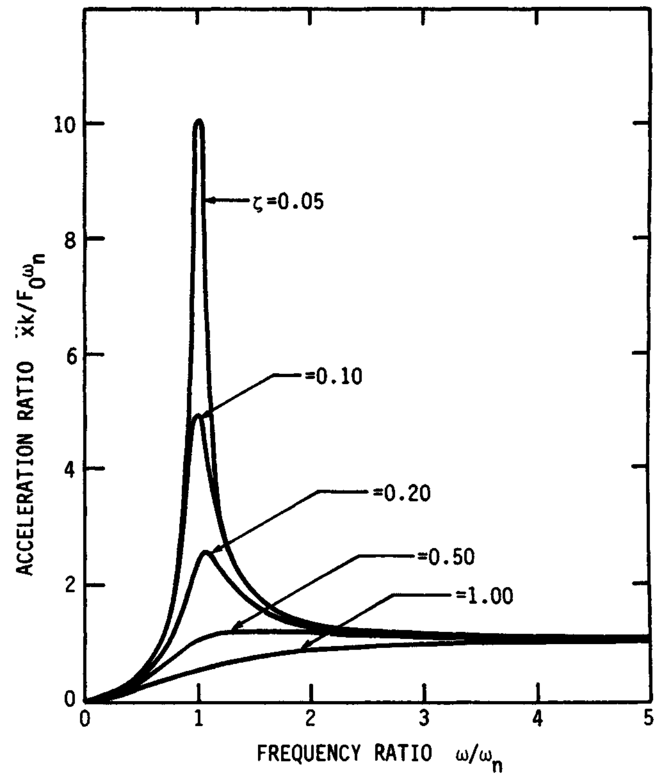

FIGURE 38.11 Acceleration frequency response.

This equation is similar to Eq. (38.25), where the force amplitude F_0 is replaced by $me\omega^2$. Hence, the steady-state solution of Eq. (38.42) is similar in form to Eq. (38.27) and is given nondimensionally as

$$\frac{x}{e}\frac{M}{m} = \frac{(\omega/\omega_n)^2 \sin(\omega t - \theta)}{[(1 - \omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2]^{1/2}} \tag{38.43}$$

where

$$\tan\theta = \frac{2\zeta\omega/\omega_n}{1 - \omega^2/\omega_n^2} \tag{38.44}$$

Note that since the excitation is proportional to ω^2, the response has an ω^2 term in the numerator and resembles the acceleration response of a system subjected to a force of constant magnitude, given by Eq. (38.35). The complete solution consists of the complementary part of the solution and is

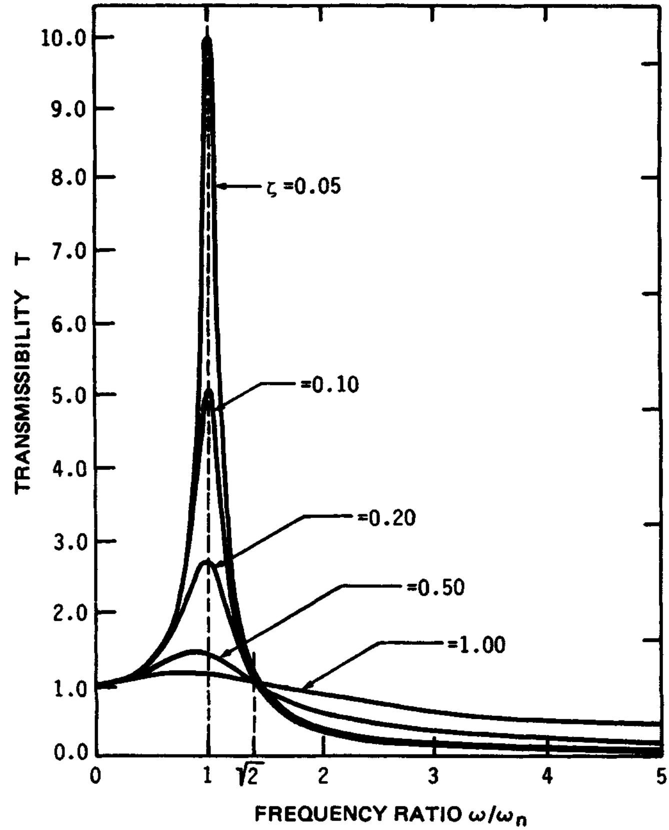

FIGURE 38.12 Transmissibility plot.

$$x = \exp -\zeta\omega_n t \,\{A \exp [(\zeta^2 - 1)^{1/2}\, \omega_n t] + B \exp [-(\zeta^2 - 1)^{1/2}\, \omega_n t]\} + \frac{me(\omega/\omega_n)^2 \sin (\omega t - \theta)}{M[(1 - \omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2]^{1/2}} \tag{38.45}$$

System Excited at the Foundation. When the system is excited at the foundation, as shown in Fig. 38.15, with a certain displacement $u(t) = U_0 \sin \omega t$, the equation of motion can be written as

$$m\ddot{x} + c(\dot{x} - \dot{u}) + k(x - u) = 0 \tag{38.46}$$

This equation can be written in the form

$$m\ddot{x} + c\dot{x} + k\dot{x} = cu_0\omega \cos \omega t + ku_0 \sin \omega t \tag{38.47}$$

$$= F_0 \sin (\omega t + \phi)$$

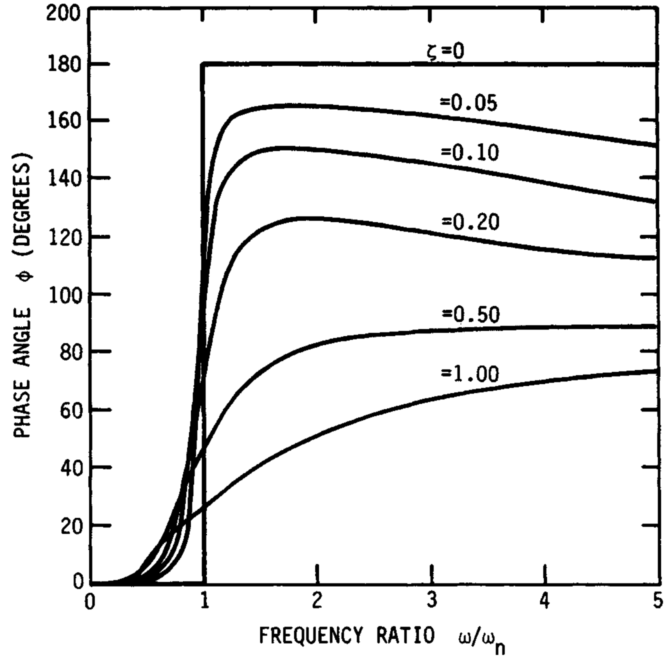

FIGURE 38.13 Phase angle between transmitted and applied forces.

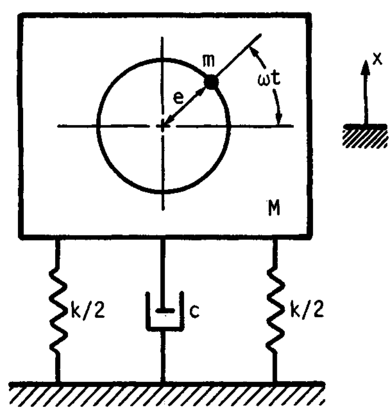

FIGURE 38.14 Dynamic system subject to unbalanced excitation.

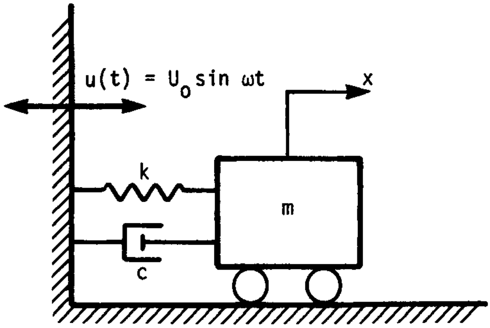

FIGURE 38.15 A base excited system.

where

$$F_0 = u_0\,(k^2 + c^2\omega^2)^{1/2} \tag{38.48}$$

and

$$\phi = \tan^{-1}\frac{k}{c\omega} \tag{38.49}$$

Equation (38.47) is identical to Eq. (38.25) except for the phase ϕ. Hence the solution is similar to that of Eq. (38.25). If the ratio of the system response to the base displacement is defined as the motion transmissibility, it will have the same form as the force transmissibility given in Eq. (38.38).

Resonance, System Bandwidth, and Q Factor. A vibrating system is said to be in resonance when the response is maximum. The displacement and acceleration responses are maximum when

$$\omega = \omega_n(1 - 2\zeta^2)^{1/2} \tag{38.50}$$

whereas velocity response is maximum when

$$\omega = \omega_n \tag{38.51}$$

In the case of an undamped system, the response is maximum when $\omega = \omega_n$, where ω_n is the frequency of free vibration of the system. For a damped system, the frequency of free oscillations or the damped natural frequency is given by

$$\omega_d = \omega_n\,(1 - \zeta^2)^{1/2} \tag{38.52}$$

In many mechanical systems, the damping is small and the resonant frequency and the damped natural frequency are approximately the same.

When the system has negligible damping, the frequency response has a sharp peak at resonance; but when the damping is large, the frequency response near resonance will be broad, as shown in Fig. 38.8. A section of the plot for a specific damping value is given in Fig. 38.16.

The Q factor is defined as

$$Q = \frac{1}{2\zeta} = R_{\max} \tag{38.53}$$

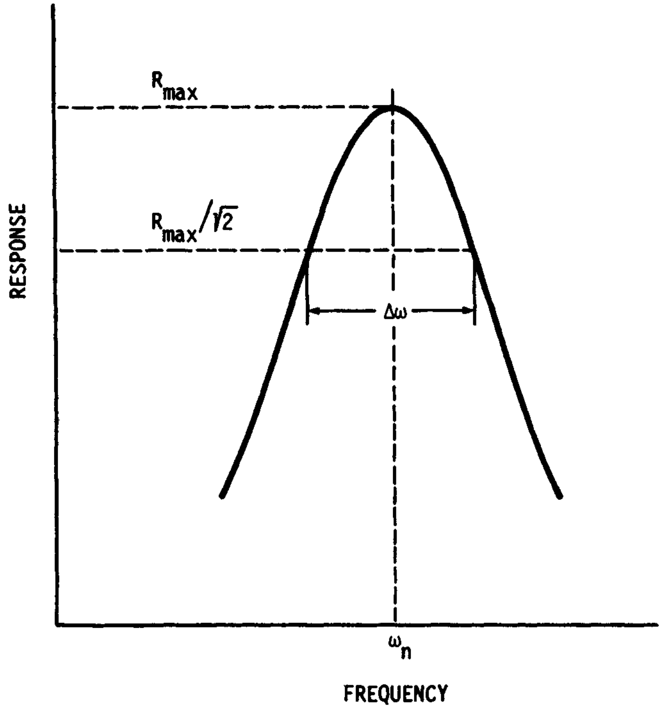

FIGURE 38.16 Resonance, bandwidth, and Q factor.

which is equal to the maximum response in physical systems with low damping. The bandwidth is defined as the width of the response curve measured at the "half-power" points, where the response is $R_{max}/\sqrt{2}$. For physical systems with $\zeta < 0.1$, the bandwidth can be approximated by

$$\Delta\omega = 2\zeta\omega_n = \frac{\omega_n}{Q} \tag{38.54}$$

Forced Vibration of Torsional Systems. In the torsional system of Fig. 38.3, if the disk is subjected to a sinusoidal external torque, the equation of motion can be written as

$$J\ddot{\theta} + c\dot{\theta} + k\theta = T_0 \sin \omega t \tag{38.55}$$

Equation (38.55) has the same form as Eq. (38.25). Hence the solution can be obtained by replacing m by J and F_0 by T_0 and by using torsional stiffness and torsional damping coefficients for k and c, respectively, in the solution of Eq. (38.25).

38.2.4 Numerical Integration of Differential Equations of Motion: Runge-Kutta Method

When the differential equation cannot be integrated in closed form, numerical methods can be employed. If the system is nonlinear or if the system excitation cannot be expressed as a simple analytical function, then the numerical method is the only recourse to obtain the system response.

The differential equation of motion of a system can be expressed in the form

$$\ddot{x} = f(x, \dot{x}, t)$$

or

$$\begin{aligned} \dot{x} &= y = F_1(x, y, t) \\ y &= f(x, \dot{x}, t) = F_2(x, y, t) \\ x_0 &= x(0) \qquad \dot{x}_0 = \dot{x}(0) \end{aligned} \tag{38.56}$$

where x_0 and $\dot{x}_0$ are the initial displacement and velocity of the system, respectively. The form of the equation is the same whether the system is linear or nonlinear.

Choose a small time interval h such that

$$t_j = jh \qquad \text{for } j = 0, 1, 2, \ldots$$

Let w_{ij} denote an approximation to $x_i\,(t_j)$ for each $j = 0, 1, 2, \ldots$ and $i = 1, 2$. For the initial conditions, set $w_{1,0} = x_0$ and $w_{2,0} = \dot{x}_0$. Obtain the approximation w_{ij+1}, given all the values of the previous steps w_{ij}, as [38.1]

$$w_{i,j+1} = w_{i,j} + \frac{1}{6}\,(k_{1,i} + 2k_{2,i} + 2k_{3,i} + k_{4,i}) \qquad i = 1, 2 \tag{38.57}$$

where

$$\begin{aligned} k_{1,i} &= hF_i(t_j + w_{1,j}, w_{2,j}) \\ k_{2,i} &= hF_i\left(t_j + \frac{h}{2}, w_{1,j} + \frac{1}{2}\,k_{1,1}, w_{2i,j} + \frac{1}{2}\,k_{1,2}\right) \\ k_{3,i} &= hF_i\left(t_j + \frac{h}{2}, w_{1,j} + \frac{1}{2}\,k_{2,1}, w_{2,j} + \frac{1}{2}\,k_{2,2}\right) \\ k_{4,i} &= hF_i(t_j + h, w_{1,i} + k_{3,1}, w_{2,i} + k_{3,2}) \qquad i = 1, 2 \end{aligned} \tag{38.58}$$

Note that $k_{1,1}$ and $k_{1,2}$ must be computed before we can obtain $k_{2,1}$.

Example. Obtain the response of a generator rotor to a short-circuit disturbance given in Fig. 38.17.

The generator shaft may be idealized as a single-degree-of-freedom system in torsion with the following values:

$$\omega_1 = 1737 \text{ cpm} = 28.95(2\pi) \text{ rad/s} = 182 \text{ rad/s}$$

$$J = 8.5428 \text{ lb} \cdot \text{in} \cdot \text{s}^2\ (25 \text{ kg} \cdot \text{m}^2)$$

$$k = 7.329 \times 10^6 \text{ lb} \cdot \text{in/rad}\ (828\,100 \text{ N} \cdot \text{m/rad})$$

Solution

$$J\ddot{\theta} + k\theta = f(t) \qquad \ddot{\theta} = \frac{k}{J}\,\theta + \frac{f(t)}{J}$$

Hence,

$$\dot{\theta} = \phi \qquad \dot{\phi} = \frac{-k}{J}\,\theta + \frac{f(t)}{J} \tag{38.59}$$

where $f(t)$ is tabulated.

Since $\omega_1 = 182$ rad/s, the period $\tau = 2\pi/182 = 0.00345$ s and the time interval h must be chosen to be around 0.005 s. Hence, tabulated values of $f(t)$ must be available for t intervals of 0.005 s, or it has to be interpolated from Fig. 38.17.

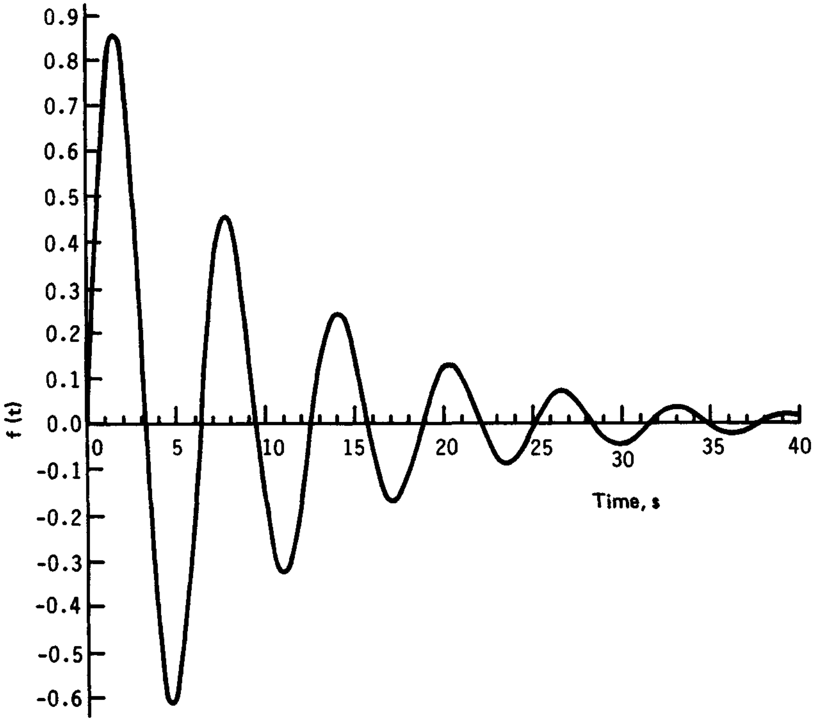

FIGURE 38.17 Short-circuit excitation form.

38.3 SYSTEMS WITH SEVERAL DEGREES OF FREEDOM

Quite often, a single-degree-of-freedom system model does not sufficiently describe the system vibrational behavior. When it is necessary to obtain information regarding the higher natural frequencies of the system, the system must be modeled as a multidegree-of-freedom system. Before discussing a system with several degrees of freedom, we present a system with two degrees of freedom, to give sufficient insight into the interaction between the degrees of freedom of the system. Such interaction can also be used to advantage in controlling the vibration.

38.3.1 System with Two Degrees of Freedom

Free Vibration. A system with two degrees of freedom is shown in Fig. 38.18. It consists of masses m_1 and m_2, stiffness coefficients k_1 and k_2, and damping coefficients c_1 and c_2. The equations of motion are

$$
\begin{aligned}
m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 + (k_1 + k_2)x_1 - c_2\dot{x}_2 - k_2x_2 = 0 \\
m_2\ddot{x}_2 + c_2\dot{x}_2 + k_2x_2 - c_2\dot{x}_1 - k_2x_1 = 0
\end{aligned}
\tag{38.60}
$$

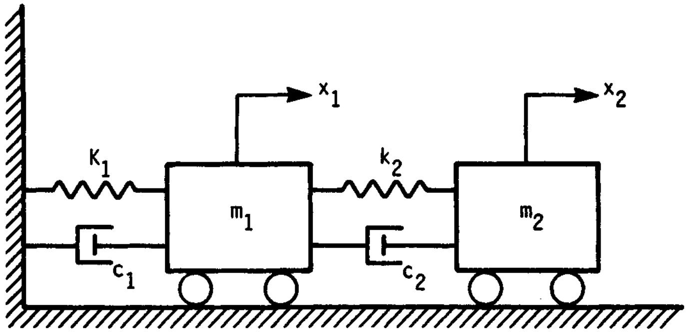

FIGURE 38.18 Two-degree-of-freedom system.

Assuming a solution of the type

$$x_1 = Ae^{st} \qquad x_2 = Be^{st} \tag{38.61}$$

and substituting into Eqs. (38.60) yield

$$\begin{aligned} [m_1 s^2 + (c_1 + c_2)s + k_1 + k_2]A - (c_2 s + k_2)B &= 0 \\ -(k_2 + c_2 s)A + (m_2 s^2 + c_2 s + k_2)B &= 0 \end{aligned} \tag{38.62}$$

Combining Eqs. (38.62), we obtain the frequency equation

$$[m_1 s^2 + (c_1 + c_2)s + k_1 + k_2]\,(m_2 s^2 + c_2 s + k_2) - (c_2 s + k_2)^2 = 0 \tag{38.63}$$

This is a fourth-degree polynomial in s, and it has four roots; hence, the complete solution will consist of four constants which can be determined from the four initial conditions $x_1, x_2, \dot{x}_1$, and $\dot{x}_2$. If damping is less than critical, oscillatory motion occurs, and all four roots of Eq. (38.63) are complex with negative real parts, in the form

$$s_{1,2} = -n_1 \pm ip_1 \qquad s_{3,4} = -n_2 \pm ip_2 \tag{38.64}$$

So the complete solution is

$$\begin{aligned} x_1 &= \exp(-n_1 t)\,(A_1 \cos p_1 t + A_2 \sin p_1 t) \\ &\qquad + \exp(-n_2 t)\,(B_1 \cos p_2 t + B_2 \sin p_2 t) \\ x_2 &= \exp(-n_1 t)\,(A_1' \cos p_1 t + A_2' \sin p_1 t) \\ &\qquad + \exp(-n_2 t)\,(B_1' \cos p_2 t + B_2' \sin p_2 t) \end{aligned} \tag{38.65}$$

Since the amplitude ratio A/B is determined by Eq. (38.62), there are only four independent constants in Eq. (38.65) which are determined by the initial conditions of the system.

Forced Vibration. Quite often an auxiliary spring-mass-damper system is added to the main system to reduce the vibration of the main system. The secondary system is called a *dynamic absorber*. Since in such cases the force acts on the main system only, consider a force $P \sin \omega t$ acting on the primary mass m. Referring to Fig. 38.18, we see that the equations of motion are

$$\begin{aligned} m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 + (k_1 + k_2)x_1 - c_2\dot{x}_2 - k_2x_2 &= P \sin \omega t \\ m_2\ddot{x}_2 + c_2\dot{x}_2 + k_2x_2 - c_2\dot{x}_1 - k_2x_1 &= 0 \end{aligned} \tag{38.66}$$

Assuming a solution of the type

$$\begin{aligned} \frac{x_1}{P/k_1} &= A_1 \cos \omega t + A_2 \sin \omega t \\ \frac{x_2}{P/k_1} &= A_3 \cos \omega t + A_4 \sin \omega t \end{aligned} \tag{38.67}$$

and substituting into Eqs. (38.66), we find that the A_i are given as

$$\begin{aligned} A_1 &= \frac{\omega_1^2[2D_1\omega\zeta_2\omega_2 - D_2(\omega_2^2 - \omega^2)]}{D_1^2 + D_2^2} \\ A_2 &= \frac{\omega_1^2[D_1(\omega_2^2 - \omega^2) + 2D_2\omega\zeta_2\omega_2]}{D_1^2 + D_2^2} \\ A_3 &= \frac{\omega_1^2(2D_1\omega\zeta_2\omega_2 - D_2\omega_2^2)}{D_1^2 + D_2^2} \\ A_4 &= \frac{\omega_1^2(D_1\omega_2^2 + 2D_2\omega\zeta_2\omega_2)}{D_1^2 + D_2^2} \end{aligned} \tag{38.68}$$

where

$$\begin{aligned} D_1 &= (\omega^2 - \omega_2^2)(\omega^2 - \omega_1^2 - \mu\omega_2^2) - 4\omega^2\zeta_2\omega_2(\zeta_1\omega_1 + \mu\zeta_2\omega_2) - \mu(\omega_2^4 - 4\omega^2\zeta_2^2\omega_2^2) \\ D_2 &= 2\omega[(\omega_2^2 - \omega^2)(\zeta_1\omega_1 + \mu\zeta_2\omega_2) + \zeta_2\omega_2(\omega_1^2 - \omega^2 + \mu\omega_2^2) - 2\mu\zeta_2\omega_2^3] \end{aligned} \tag{38.69}$$

$$\begin{aligned} \omega_1^2 &= \frac{k_1}{m_1} \qquad \omega_2^2 = \frac{k_2}{m_2} \\ \zeta_1 &= \frac{c_1}{2m_1\omega_1} \qquad \zeta_2 = \frac{c_2}{2m_2\omega_2} \\ \mu &= \frac{m_2}{m_1} \end{aligned} \tag{38.70}$$

Responses may also be written in the form

$$x_1 = B_1 \sin(\omega t - \theta_1) \qquad x_2 = B_2 \sin(\omega t - \theta_2) \tag{38.71}$$

where

$$B_1 = (A_1^2 + A_2^2)^{1/2} \qquad B_2 = (A_3^2 + A_4^2)^{1/2}$$
$$\tan\theta_1 = -\frac{A_1}{A_2} \qquad \tan\theta_2 = -\frac{A_3}{A_4} \tag{38.72}$$

Here θ_1 and θ_2 are the phase angles by which the responses of masses m_1 and m_2, respectively, will lag behind the applied force. The response amplitudes B_1 and B_2 are plotted in Figs. 38.19 and 38.20, respectively. The amplitude B_1 has a minimum between ω_1 and ω_2.

The equations of motion for torsional systems with 2 degrees of freedom have the same form as Eqs. (38.60) and (38.66). The solution will also be similar and will exhibit the same characteristics as discussed earlier.

38.3.2 Multidegree-of-Freedom Systems

In many applications, it is necessary to know several higher modes of a vibrating system and evaluate the vibration response. Here, the elastic system has to be treated as one with distributed mass and elasticity. This is possible for simple elements such as beams, plates, or shells of regular geometry. However, when the structural system is complex, it may be modeled as a multidegree-of-freedom discrete

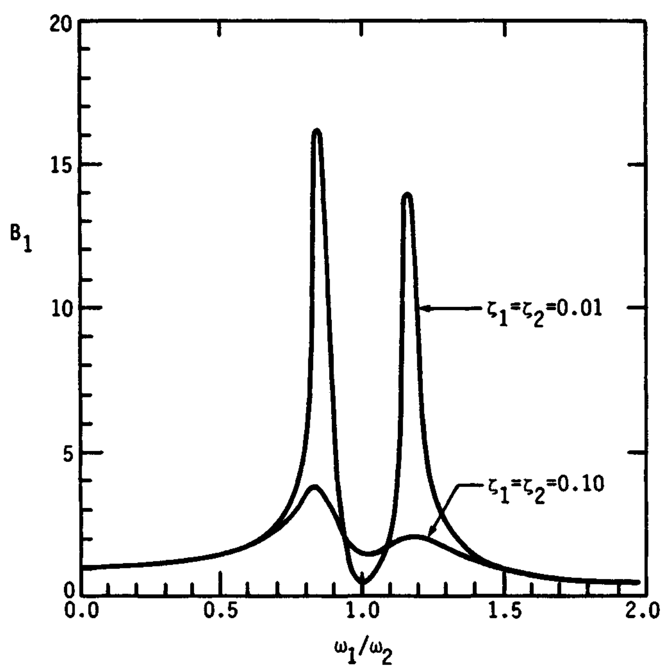

FIGURE 38.19 Amplitude frequency response of the mass of a two-freedom system subject to forced excitation.

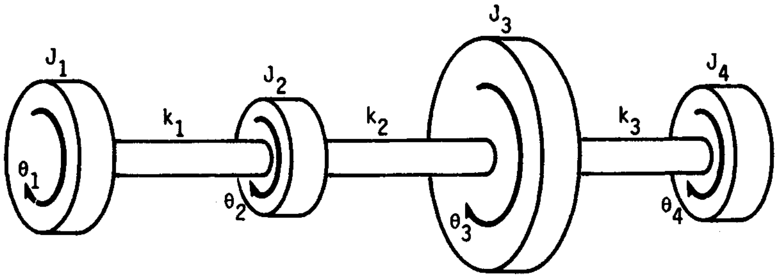

FIGURE 38.20 Torsional system with four freedoms.

system by concentrating its mass and stiffness properties at a number of locations on the structure.

The number of degrees of freedom of a structure is the number of independent coordinates needed to describe the configuration of the structure. In a lumped-mass model, if motion along only one direction is considered, the number of degrees of freedom is equal to the number of masses; and if motion in a plane is of interest, the number of degrees of freedom will equal twice the number of lumped masses.

Holzer Method. When an undamped torsional system consisting of several disks connected by shafts vibrates freely in one of its natural frequencies, it does not need any external torque to maintain the vibration. In Holzer's method, this fact is used to calculate the natural frequencies and natural modes of a vibrating system. Figure 38.20 shows a torsional system with several disks connected by shafts. In this procedure, an initial value is assumed for the natural frequency, and a unit amplitude is specified at one end. The resulting torques and angular displacements are progressively calculated from disk to disk and carried to the other end. If the resulting torque and displacement at the other end are compatible with boundary conditions, the initial assumed value for the natural frequency is a correct natural frequency; if not, the whole procedure is repeated with another value for the natural frequency until the boundary conditions are satisfied. For a frequency ω and $\theta_1 = 1$, the corresponding inertial torque of the first disk in Fig. 38.20 is

$$T_1 = -J_1\ddot{\theta}_1 = J_1\omega^2\theta_1 \tag{38.73}$$

This torque is transmitted to disk 2 through shaft 1; hence,

$$T_1 = J_1\omega^2\theta_1 = k_1\,(\theta_1 - \theta_2) \tag{38.74}$$

which relates θ_2 and θ_1. The inertial torque of the second disk is $J_2\omega^2\theta_2$, and the sum of the inertial torques of disk 1 and disk 2 is transmitted to disk 3 through shaft 2, which gives

$$J_1\omega^2\theta_1 + J_2\omega^2\theta_2 = k_2(\theta_2 - \theta_3) \tag{38.75}$$

Continuing this process, we see the torque at the far end is the combined inertial torques of all the disks and is given by

$$T = \sum_{i=1}^{n} J_i\omega^2\theta_i \tag{38.76}$$

where n is the total number of disks. If the disk is free at that end, the total torque T should vanish. Hence, the frequency ω which makes T zero at the far end is a natural frequency.

Example. Determine the natural frequencies and mode shapes of a torsional system consisting of three disks connected by two shafts.

$$J_1 = 1.7086 \times 10^4 \text{ lb} \cdot \text{in} \cdot \text{s}^2 \qquad (5 \text{ kg} \cdot \text{m}^2)$$

$$J_2 = 3.7588 \times 10^4 \text{ lb} \cdot \text{in} \cdot \text{s}^2 \qquad (11 \text{ kg} \cdot \text{m}^2)$$

$$J_3 = 3.4171 \times 10^4 \text{ lb} \cdot \text{in} \cdot \text{s}^2 \qquad (10 \text{ kg} \cdot \text{m}^2)$$

$$k_1 = 8.8504 \times 10^5 \text{ lb} \cdot \text{in/rad} \qquad (1 \times 10^5 \text{ rad})$$

$$k_2 = 1.7701 \times 10^6 \text{ lb} \cdot \text{in/rad} \qquad (2 \times 10^5 \text{ rad})$$

Solution. Holzer's procedure can be carried out in a tabulated form as shown in Table 38.1. Two trials are shown in Table 38.1. The calculation can be carried out for more values of ω, and the resulting T_3 can be plotted versus ω, as shown in Fig. 38.21. The frequencies at which $T_3 = 0$ are then the natural frequencies of the system. Better approximation can be obtained by employing the method of false position [38.1]; if $\omega(+)$ and $\omega(-)$ are the frequencies when the torque has corresponding values of $T_3(+)$ and $T_3(-)$, the natural frequency can be obtained by

TABLE 38.1 Holzer's Procedure

Frequency, rad/s	Station					
	1		2		3	
ω	θ	$T = J\omega^2\theta$	$\theta = 1 - \frac{T}{K}$	$T = T + J\omega^2\theta$	$\theta = \theta - \frac{T}{K}$	$T = T + J\omega^2\theta$
	1	1 1 1	2 1 1	2 1 2 2	3 2 2 2	3 2 3 3
10	1.0	4 425.2 (500.0)	0.995	1 412.0 (1 594.5)	0.987	22 847.0 (2 581.5)
20	1.0	17 701.0 (2 000.0)	0.980	55 864.0 (6 312.0)	0.948	89 442.0 (10 106.0)

$$\omega = \frac{\omega(-)T_3(+) - \omega(+)T_3(-)}{T_3(+) - T_3(-)}$$

The mode shape corresponding to a natural frequency can be obtained by recalculating the values of θ_1, θ_2, and θ_3 in the Holzer table. In the example, the first natural frequency is $\omega_1 = 141.4214$ rad/s, and the corresponding mode shape is

$$\{\theta_1, \theta_2, \theta_3\} = \{1.0, 0.0, -0.5\}$$

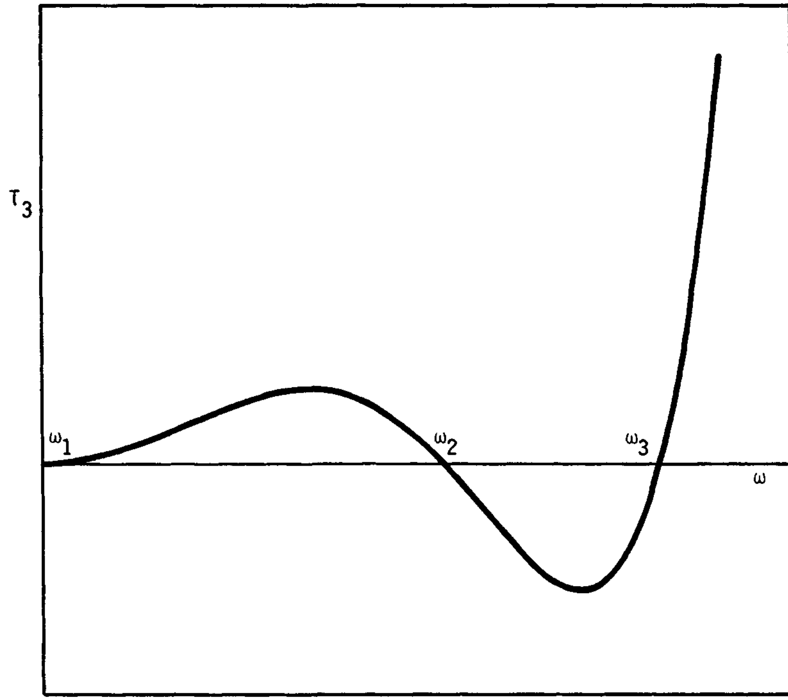

FIGURE 38.21 Variation of end torque with assumed natural frequency.

Geared Systems. When a shaft transmits torque to another through a gear drive of speed ratio n, it is necessary to reduce the geared torsional system to an equivalent single-shaft system to find its natural frequency. The moments of inertia and the stiffness of the equivalent system are obtained through a consideration of the kinetic and potential energies of the system.

Consider the geared torsional system in Fig. 38.22*a*. The speed of the second shaft is $\dot{\theta}_2 = n\dot{\theta}_1$. Assuming massless gears, we see that the kinetic energy of the system is

$$T = \frac{1}{2} J_1 \dot{\theta}_1^2 + \frac{1}{2} J_2 n^2 \dot{\theta}_1^2 \tag{38.77}$$

Thus the equivalent mass moment of inertia of disk 2 referred to shaft 1 is $n^2 J_2$. If disks 1 and 2 are clamped and a torque is applied to gear 1, rotating it through an angle θ_1, there will be deformations in both shafts 1 and 2. Gear 2 will rotate through an angle $\theta_2 = n\theta_1$. The potential energy stored in the two shafts is

$$U = \frac{1}{2} k_1 \theta_1^2 + \frac{1}{2} k_2 n^2 \theta_1^2 \tag{38.78}$$

Hence the equivalent stiffness of shaft 2 referred to shaft 1 is $n^2 k_2$. The equivalent torsional system is shown in Fig. 38.22*b*, where the stiffness and inertia of one side of the system are multiplied by the square of the speed ratio to obtain the corresponding equivalent values for Holzer calculations.

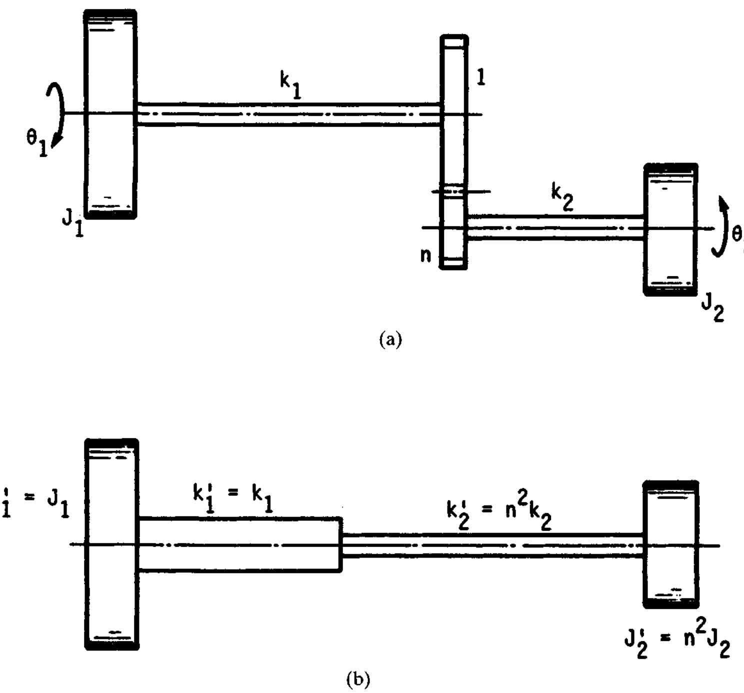

FIGURE 38.22 (*a*) Geared shaft disk system; (*b*) equivalent torsional system.

38.3.3 Continuous Systems

Engineering structures, in general, have distributed mass and elasticity. Such structures can be treated as multidegree-of-freedom systems by lumping their masses at certain locations and connecting them by representative spring elements. However, it is necessary to consider several such lumped masses and springs to get sufficiently accurate values for the natural frequencies. If only the fundamental natural frequency or the first few natural frequencies are of interest, it is convenient to use some approximate methods based on energy formulations discussed here.

Rayleigh Method. This method can give the natural frequency of a structure of any specific mode of vibration. A deflection shape satisfying the geometric boundary conditions has to be assumed initially. If the natural frequency of the fundamental mode of vibration is of interest, then a good approximation would be the static deflection shape. For a harmonic motion, the maximum kinetic energy of a structure can be written in the form

$$T_{\max} = \omega^2 C_1 \tag{38.79}$$

where C_1 depends on the assumed deflection shape. The maximum potential energy is of the form

$$U_{\max} = C_2 \tag{38.80}$$

Neglecting damping, we see that the maximum kinetic energy must be equal to the maximum potential energy. Hence, the natural frequency is

$$\omega^2 = \frac{C_2}{C_1} \tag{38.81}$$

This estimate will always be higher than the true natural frequency.

Example. Determine the fundamental natural frequency of a uniform cantilever beam of length L supporting a disk of mass M and diametral mass moment of inertia I_d, as shown in Fig. 38.23. The modulus of elasticity of the beam material is E, the mass moment of inertia of the cross section is I, and the mass per unit length of the beam is m.

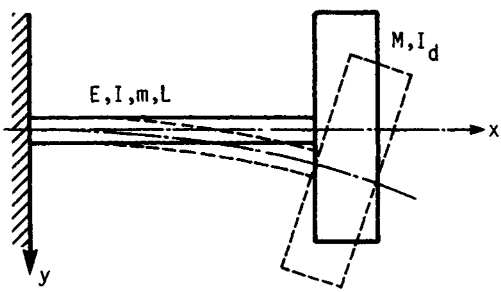

FIGURE 38.23 Cantilever with end mass.

Solution. The deflection shape may be assumed to be $y(x) = Cx^2$, which satisfies the geometric boundary conditions of zero deflection and zero slope at $x = 0$. The maximum kinetic energy of the structure for harmonic vibration is

$$T_{\max} = \frac{1}{2} m\omega^2 \int_0^L y^2(x)\, dx + \frac{1}{2} M\omega^2 y^2(L) + \frac{1}{2} I_d \omega^2 y'^2(L)$$

The strain energy is given by

$$U_{\max} = \frac{1}{2} EI \int_0^L y''^2(x)\, dx$$

Substituting $y(x) = Cx^2$ in the above expressions and equating $T_{\max} = U_{\max}$, we find the natural frequency

$$\omega^2 = \frac{20EI}{mL^4 + 5ML^3 + 20I_dL}$$

In the absence of the disk, $\omega = 4.47\ (EI/mL^4)^{1/2}$. By comparing this to the exact result $\omega = 3.52\ (EI/mL^4)^{1/2}$, the error in the approximation is error $= 0.95\ (EI/mL^4)^{1/2}$, or 26.9 percent.

This error can be reduced by obtaining the strain energy by using a different method. The shear at any section is obtained by integrating the inertial loading from the free end as

$$V(\xi) = \left[\omega^2 \int_\xi^L m(\xi) y(\xi)\, d\xi\right] + M\omega^2 y(L)$$

$$= \frac{1}{3}\omega^2 mc(L^3 - \xi^3) + M\omega^2 cL^2$$

and the moment at any point x is

$$M(x) = \int_x^L V(\xi)\, d\xi + I_d y'(L) = \frac{1}{12}\,\omega^2 mc(3L^4 - 4L^3 x + x^4) + M\omega^2 cL^2(L - x) + 2I_d cL$$

The strain energy is then

$$U_{\max} = \frac{1}{2}\int_0^L \frac{M^2(x)}{EI}\, dx$$

When the disk is absent,

$$U_{\max} = \frac{\omega^4}{2EI}\,\frac{m^2c^2}{144}\,\frac{312}{135}\,L^9$$

With $T_{\max}$ and $U_{\max}$ equated, the natural frequency $\omega = 3.53\ (EI/ML^4)^{1/2}$ has an error of only 0.28 percent.

38.4 VIBRATION ISOLATION

Often machines and components which exhibit vibrations have to be mounted in locations where vibrations may not be desirable. Then the machine has to be isolated properly so that it does not transmit vibrations.

38.4.1 Transmissibility

Active Isolation and Transmissibility. From Eq. (38.38), the force transmissibility, which is the magnitude of the ratio of the force transmitted to the force applied, is given by

$$T = \left[\frac{1 + (2\zeta\omega/\omega_n)^2}{(1 - \omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2}\right]^{1/2} \tag{38.82}$$

Equation (38.82) is plotted in Fig. 38.12 for different values of ζ. All the curves cross at $\omega/\omega_n = \sqrt{2}$. For $\omega/\omega_n > \sqrt{2}$, transmissibility, although below unity, increases with an increase in damping, contrary to normal expectations. At higher frequencies, transmissibility goes to zero.

Since the force amplitude $me\omega^2$ in the case of an unbalanced machine is dependent upon the operating speed of the machine, transmissibility can be defined as

$$T = \frac{F_T}{me\omega^2} = \left(\frac{\omega}{\omega_n}\right)^2 \left[\frac{1 + (2\zeta\omega/\omega_n)^2}{(1 - \omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2}\right]^{1/2} \tag{38.83}$$

where F_T is the amplitude of the transmitted force.

Equation (38.83) is plotted in Fig. 38.24. Transmissibility starts from zero at zero operating frequency, and curves for different damping ratios cross at $\omega/\omega_n = \sqrt{2}$. For higher values of operating speed, transmissibility increases indefinitely with frequency.

Passive Isolation. When a sensitive instrument is isolated from a vibrating foundation, it is called *passive isolation*. Consider the system shown in Fig. 38.15, where the base has a motion $u = U_0 \sin \omega t$. The equation of motion of the system is given in Eq. (38.46).

The ratio of the response and excitation amplitudes is

$$\frac{X}{X_f} = \frac{k + i\omega c}{k - m\omega^2 + i\omega c} \tag{38.84}$$

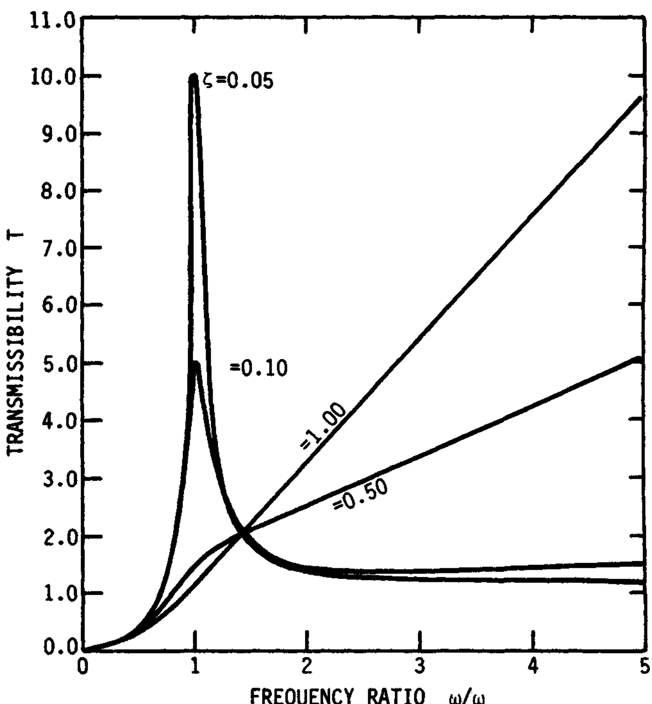

FIGURE 38.24 Transmissibility of a system under unbalanced excitation.

Since we are interested in the motion transmissibility in the case of passive isolation, Eq. (38.84) gives the transmissibility T, which can be put in terms of nondimensional parameters, as

$$T = \frac{X}{U_0} \left[\frac{1 + (2\zeta\omega/\omega_n)^2}{(1 - \omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2} \right]^{1/2} \tag{38.85}$$

Equation (38.85) is identical to the force transmissibility in the case of active isolation given in Eq. (38.82).

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